

DPNU-96-06
(Revised)
January 1996

Slow Relaxation at Critical Point of Second Order Phase Transition in a Highly Chaotic Hamiltonian System

Yoshiyuki Y. YAMAGUCHI ¹

Department of Physics, School of Science,
Nagoya University, Nagoya, 464-01, Japan

Abstract

Temporal evolutions toward thermal equilibria are numerically investigated in a Hamiltonian system with many degrees of freedom which has second order phase transition. Relaxation processes are studied through local order parameter, and slow relaxations of power type are observed at the critical energy of phase transition for some initial conditions. Numerical results are compared with results of a phenomenological theory of statistical mechanics. At the critical energy, the maximum Lyapunov exponent takes the largest value. Temporal evolutions and probability distributions of local Lyapunov exponents show that the system is highly chaotic rather than weakly chaotic at the critical energy. Consequently theories for perturbed systems may not be applied to the system at the critical energy in order to explain the slow relaxation of power type.

1 Introduction

In recent years some researchers have investigated Hamiltonian systems with many degrees of freedom [1] (and references therein). They are interested in statistical mechanics since Hamiltonian dynamics is the basis of it. They mainly have considered relaxation processes to equilibrium states because we can observe temporal evolutions of systems directly from Hamiltonian dynamics.

In both the one-dimensional classical Φ^4 lattice model and FPU β model, they observed slow and fast relaxation processes in low and high energy regions, respectively [2][3]. The slow and fast relaxation processes occur in weakly and highly chaotic systems, respectively [4]. Here the word of weakly chaotic system means that theories for perturbed systems may be applied for the system. For instance, if we add small non-integrable perturbation to a completely integrable

¹e-mail: yamaguchi@allegro.phys.nagoya-u.ac.jp

system, whose phase space consist of tori, almost all tori survive under the small perturbation. That is a result of Kolmogorov-Arnold-Moser (KAM) theorem [5]. Intermittency between chaotic and regular-like motions is also a characteristic phenomenon which occurs in perturbed systems. Arnold diffusion , [6] Nekhoroshev time[7] and other theories exist for perturbed systems. Using theories and properties of weakly chaotic systems, above-mentioned slow relaxation, which occurs in weakly chaotic systems, was discussed[2].

Properties of weakly chaotic systems have been well investigated both theoretically and numerically. Contrary to weakly chaotic systems, few studies exist for highly chaotic systems, and to study properties of highly chaotic systems whose dynamics can not reduced to Markovian process is a left important problem of classical Hamiltonian systems.

We expect it is useful to consider relaxation processes in a Hamiltonian system having second order phase transition in order to study properties of highly chaotic systems. The reason is as follows. The system seems highly chaotic near the critical point because fluctuation anomalously increases. Nevertheless, according to a phenomenological theory of statistical mechanics, van Hove theory, slow relaxation of power type of order parameter appears at the critical point. That means a certain structure exists in phase space.

In this way we understand that critical phenomenon is interesting as a problem in dynamics, as well as in statistical mechanics.

In this paper, we investigate the following things. First, as a problem of statistical mechanics, we verify appearance of the slow relaxation of order parameter at the critical point even when we use a dynamical method, and we compare results of dynamics with ones of phenomenological theory. Second, as a problem of dynamics, we confirm that the system is highly chaotic at the critical point, and we suggest that critical phenomenon is a useful phenomenon to study properties of highly chaotic systems.

We introduce a Hamiltonian system in which second order phase transition occurs, and numerically integrate the equations of motion. The model Hamiltonian is as follows [8]

$$H(q, p) = \sum_{i=1}^N \frac{1}{2} p_i^2 + U(q), \quad (1)$$

$$U(q) = \frac{1}{2N} \sum_{i,j=1}^N (1 - \cos(q_i - q_j)). \quad (2)$$

Equations of motion are derived from the Hamiltonian as follows

$$\frac{dq_i}{dt} = p_i \quad (3)$$

$$\frac{dp_i}{dt} = -\frac{1}{N} \sum_{j=1}^N \sin(q_i - q_j), \quad (4)$$

and temporal evolutions of the system are yielded from the equations of motion. Each particle moves on the unit circle and interacts with all the others. The q_i 's are phase of particles and the p_i 's are canonical conjugated momenta. According to statistical mechanics, the critical energy E_c is $E_c/N = 0.75$ [9], where N represents degrees of freedom.

We define an order parameter parameter M of this system as

$$M = \frac{1}{L} \int_0^L dt M(t) \quad (5)$$

and

$$M(t) = \|\vec{M}(t)\|, \quad \vec{M}(t) = \left(\frac{1}{N} \sum_{i=1}^N \cos q_i(t), \frac{1}{N} \sum_{i=1}^N \sin q_i(t) \right), \quad (6)$$

where t and $\|\vec{M}(t)\|$ represent time and absolute value of $\vec{M}(t)$, respectively. From the definition, $0 \leq M, M(t) \leq 1$. When all particles distribute uniformly on the unit circle at each time, then $M = 0$. On the other hand, when particles form the cluster, which is the only one in this system [10], then $M \sim 1$. For this system, Antoni and Ruffo showed that energy dependence of order parameter obtained from numerical experiments agrees with a result of statistical mechanics [9].

We observe relaxation processes, that are temporal evolutions of $M(t)$, toward equilibria using a local order parameter $M_\tau(n)$, which is defined as

$$M_\tau(n) = \frac{1}{\tau} \int_{(n-1)\tau}^{n\tau} dt M(t). \quad (7)$$

The system Eq.(1) is integrable in the limit $E/N \rightarrow 0$ and $E/N \rightarrow \infty$. For the former and the latter limit the system is approximated as harmonic oscillators and free particles, respectively. To explain the case of the latter limit, we show the following expression of the potential term $U(q)$:

$$U(q) = \frac{N}{2} - \frac{1}{2N} \sum_{i,j=1}^N (\cos q_i \cos q_j + \sin q_i \sin q_j) \quad (8)$$

$$= \frac{N}{2} (1 - M(t)^2), \quad 0 \leq M(t) \leq 1 \quad (9)$$

(cf. Eq.(6)). Hence $U(q)$ has the minimum and the maximum value

$$0 \leq U(q)/N \leq 1/2. \quad (10)$$

We can therefore neglect the potential term when the value of E/N is large enough, then the system consists of free particles. Note that $\lambda_1 \rightarrow 0$ when $E/N \rightarrow 0$ and $E/N \rightarrow \infty$ (see inset of Fig.6).

To compare our numerical results with results of a phenomenological theory of statistical mechanics, we introduce a phenomenological theory: van Hove theory.

We assume temporal evolution of order parameter $M(t)$ is determined by gradient of Landau's free energy $F(M)$, that is,

$$\frac{dM(t)}{dt} \propto -\frac{\partial F(M)}{\partial M}, \quad (11)$$

$$F(M) = a'(T - T_c)M^2 + b(T)M^4, \quad (a' > 0). \quad (12)$$

Note that relaxations to equilibria occur at once if states are non-equilibria ($\partial F(M)/\partial M \neq 0$). From the solution of Eq.(11), we obtain the following temporal evolution of $M(t)$

$$M(t) \sim \begin{cases} \exp(-t/\tau(T)) & \text{(when } T \gtrless T_c\text{)} \\ t^{-1/2} & \text{(when } T = T_c\text{)} \end{cases}, \quad (13)$$

where $\tau(T)$ is called relaxation time which depends on temperature T and diverges at critical temperature T_c , and Eq.(13) is approximately correct for $t \gg \tau(T)$ when $T \gtrless T_c$. According to this theory, relaxation of power type appears if and only if systems are just on critical points.

To integrate the equations of motion which are derived from the Hamiltonian of Eq.(1), we use forth order symplectic integrators [11] with fixed time slice $\Delta t = 0.01$ ($\Delta t = 0.001$ for a part of result in Fig.6) which keep symplectic properties of Hamiltonian systems exactly and total energy accurately. Relative errors of total energy $\Delta E/E$ are less than $O(10^{-7})$ for considered energy region. The value of τ in Eq.(7) is fixed at 10, namely 1000 steps except for the case given special comments. As initial conditions we choose random variables which follow Gaussian distribution. To see relaxation of order parameter, we take small values for q_i 's and set $M \sim 1$. Scales of p_i 's are defined from energy. In the model Eq.(1) total momentum is conserved, and we set total momentum is equal to zero.

This paper is organized as follows. In Section 2 results of numerical experiments are reported. At first, slow relaxation of power type is shown, then we investigate dependences on degrees of freedom and initial condition for appearance of the slow relaxation. After that we compare our numerical results with results of van Hove theory. Next we confirm that the system is highly chaotic at the critical energy with the aid of temporal evolutions and probability distributions of local order parameter. Section 3 is devoted to summary and discussions.

2 Results of simulations

In this section numerical results are reported for the Hamiltonian system Eq.(1). In Sec.2.1 we find that slow relaxation of power type appears at the critical energy for a certain degrees of freedom and initial condition. We observe dependences on

degrees of freedom and initial condition for appearance of the slow relaxation in Sec.2.2 and Sec.2.3, respectively. Section 2.4 is devoted to compare our dynamical results and results of a phenomenological theory of statistical mechanics. After that we investigate dynamical properties of the system, in particular, at the critical energy. Energy dependence of maximum Lyapunov exponent is shown in Sec.2.5. Then, in Sec.2.6, we confirm that the system is highly chaotic at the critical energy. That is, the slow relaxation of power type may not be explained by theories for perturbed systems. We study the structure of phase space in Sec.2.7, and suggest phase space is uniform at the critical energy for the property of instability.

2.1 Slow relaxation at the critical energy

We show a energy dependence of order parameter M in Fig.1. The solid line in the figure represents the curve obtained from theory of statistical mechanics. Using the saddle point method, it is described as simultaneous implicit functions [9]:

$$M = \frac{I_1}{I_0}(\beta M), \quad (14)$$

$$E/N = \frac{1}{2\beta} + \frac{1}{2}(1 - M^2), \quad (15)$$

where I_0 and I_1 are Bessel functions of 0-th and 1-st order, respectively. Then we find that results of numerical experiments fit the theoretical curve, and the results are reproduction of results of Antoni and Ruffo[9]. Although we find differences between the results and theory, we can understand causes of the differences as follows. For high energy part ($E/N > 1$), $M(t)$ fluctuates around zero, which is the minimum value of $M(t)$, since the system is approximated as free particles, and $q_i(t)$'s takes random variables. Then $M(t)$ is estimated around $O(1/\sqrt{N})$ from central limit theorem. That is confirmed from N dependence of the values of M in this part. For middle energy part ($E \sim E_c$), in addition to the cause of differences in high energy part, there is another cause. That is lack of time L in which we take time average of $M(t)$ because, according to van Hove theory, relaxation time increases as energy goes to critical value.

Here let us investigate relaxation of $M_\tau(n)$ defined as Eq.(7). We report results when $E/N = 0.5, 0.75 (= E_c/N)$ and 1 in Fig.2. The insets of Fig.2 represent the following quantity as a function of time t

$$M(t; t_0) = \frac{1}{t - t_0} \int_{t_0}^t dt M(t). \quad (16)$$

This quantity represents time average of $M(t)$ from t_0 to t , and reduces fluctuation $\xi(t)$ when temporal evolution of $M(t)$ is as follows

$$M(t) = (t - t_0)^{-x} + \xi(t). \quad (17)$$

For Fig.2(b) we set $t_0 = 1000$ since slow relaxation of $M_\tau(n)$ starts around the t_0 . For Figs.2(a) and (c) we set $t_0 = 0$. From Fig.2, we confirm the relaxation is power type when $E = E_c$ for the initial condition. This result agrees with a result of van Hove theory. The slow relaxation finishes in finite time since $M(t)$ takes degree of $O(1/\sqrt{N})$ for $t \rightarrow \infty$.

2.2 N dependence of slow relaxation

We observed slow relaxation of power type when $N = 80$. Now we show that the slow relaxation appears even if we change degrees of freedom. When $N = 40$ the slow relaxation of power type is shown in Fig.3. Inset of the figure represents $M(t; t_0)$ (cf. Eq.(16) against t again, where we set $t_0 = 5000$. However we have not observed the slow relaxation when $N > 80$, and we will discuss the reason and behaviors of the system in thermodynamic limit in Sec.3

2.3 Initial condition dependence

In the previous section, we investigated dependence on degrees of freedom N . Since we treat dynamical system we must study dependence on initial conditions for relaxation of $M_\tau(n)$, too. A problem is whether slow relaxation always appears or not when $E = E_c$. Figure 4 shows results when $N = 80, 200$ and 1000 . Exponential relaxations become clearer and clearer as N increases hence slow relaxation of power type does not always appear. A rate of initial conditions which yield slow relaxation may be low since only one initial condition yields slow relaxation in ten initial conditions when $E = E_c$ and $N = 80$.

2.4 Comparing with a phenomenological theory

Up to now we observed appearance of slow relaxation of power type at the critical energy for some initial conditions although the slow relaxation does not always appears for any initial conditions even if $E = E_c$. The appearance of the slow relaxation is a agreement with a result of a phenomenological theory of statistical mechanics: van Hove theory. However a disagreement also exists between our results and results of the phenomenological theory, and here we discuss the disagreement.

We found there are two time regions for slow relaxation process in Figs.2(b) and 3. One is the region in which the system stays in non-equilibrium states (flat region), the other is the region in which slow relaxation of power type occurs (relaxation region). The existence of the flat region is different from theory of van Hove, which says systems in non-equilibrium states goes toward equilibrium states immediately (cf. Eq.(11)). Our dynamical results are therefore an example of a disagreement with a result of van Hove theory.

The flat region means the existence of “induction period” and that reminds induction phenomenon, which means that a nearly periodic motion starts to behave as a non-periodic motion after a certain amount of time, which is called induction period [12][13][14]. However the phenomenon appearing in Figs.2(b) and 3 is different from induction phenomenon since the motion in phase space is not nearly periodic both before and after the onset of slow relaxation. To confirm that we show average of power spectra of momenta p_j ’s ($j = 1, 2, \dots, N_0$), $S_p(f)$, when $N = 80$ in Fig.5. The quantity $S_p(f)$ is defined as

$$S_p(f) = \frac{1}{N_0} \sum_{j=1}^{N_0} S_j(f), \quad (18)$$

$$S_j(f) = \text{“power spectrum of time series of } p_j(t) \text{”}. \quad (19)$$

Here we selected N_0 momenta from N momenta to calculate $S_p(f)$, and we set $N_0 = 40$. In low frequency region the power spectrum is growing up hence the motion is not nearly periodic.

2.5 Lyapunov exponent

We have investigated slow relaxation of power type using dynamical method. Now we study dynamical properties of the slow relaxation and structure of phase space in which the slow relaxation occurs.

As the first step, we investigate an energy dependence of the maximum Lyapunov exponent λ_1 (hereafter we call this Lyapunov exponent simply) which measures instability of orbits. Existence of positive Lyapunov exponent means a sample orbit has instability. Results of numerical experiments are reported in Fig.6 and we find Lyapunov exponent takes the largest value at $E_c/N (= 0.75)$ for all sample orbits when $N = 40, 80$ and 200 .

We can understand why value of Lyapunov exponent takes the maximum at E_c as follows. The system is integrable for low and high energy limit since it is harmonic oscillators and free particles, respectively. Thus $\lambda_1 \rightarrow 0$ when $E/N \rightarrow 0$ and $E/N \rightarrow \infty$. Integrabilities of those two limits break as energy goes away from zero or infinity, and the two integrabilities balance at the critical point. Consequently Lyapunov exponent is the maximum at the critical point.

Butera et al. also investigated Lyapunov exponent for a Hamiltonian system which is a two-dimensional system of coupled rotators [15]. They found a “knee-like” shape in the graph of Lyapunov exponent against temperature and the “knee” is at the critical temperature of Kosterlitz-Thouless (KT) phase transition [16] although the “knee” does not appear at the largest value of Lyapunov exponent. Figure 6 shows a “knee-like” shape and the “knee” is at the critical energy where λ_1 is the largest. This fact suggests that degree of instability has some relations to phase transition.

2.6 Highly chaotic property at the critical energy

We found that Lyapunov exponent takes the largest value at the critical energy hence we expect the system is highly chaotic at the critical energy. In this section we confirm that the system is highly chaotic at the critical energy from temporal evolutions and probability distributions of local Lyapunov exponents. The definition of the local Lyapunov exponent is as follows

$$\lambda_\tau(n) = \frac{1}{\tau} \int_{(n-1)\tau}^{n\tau} dt \lambda_1(t), \quad (20)$$

$$\lambda_1(t) = \frac{d}{dt} \log(\|X(t)\|) \quad (21)$$

where $X(t)$ is a $2N$ -dimensional tangent vector at time t which obeys linearized equations of motion. The local Lyapunov exponent $\lambda_\tau(n)$ indicates instability of orbits in the time interval $[(n-1)\tau, n\tau]$. Orbits are instable if $\lambda_\tau(n) > 0$ in the time interval.

Figure 7 shows temporal evolutions of local Lyapunov exponents for $E/N = 0.2$ and $0.75 (= E_c/N)$. The initial condition of Fig.7(b) yields slow relaxation of power type (cf. Fig.2(b)). When $E/N = 0.2$, typical intermittency is found. On the other hand, when $E/N = 0.75 = E_c/N$, behavior of the temporal evolution is different from the case of $E/N = 0.2$ hence motion is not intermittency for the initial condition. Moreover local Lyapunov exponents do not take near zero, hence few tori, if any, exist in phase space. Consequently the system is highly chaotic.

To make sure of the difference between the two time series of local Lyapunov exponent, we show probability distributions of local Lyapunov exponents in Fig.8. If the system is weakly chaotic and intermittency occurs, a probability distribution of local Lyapunov exponents should have a peak near zero. However, according to Fig.8, the distributions have not peaks near zero at $E = E_c$. Thus we understand the system is different from weakly chaotic system at $E = E_c$. Furthermore, at $E = E_c$, motion is not reduced to Markovian process since relaxation of power type is observed.

2.7 Uniformity of phase space at the critical energy

We observed slow relaxation of power type in highly chaotic system. If the system is weakly chaotic, we can understand the cause of the slow relaxation from the structure of phase space using theories for perturbed systems. However the slow relaxation occurs in highly chaotic system, thus we can not use the theories and we must newly investigate the structure of phase space to understand the cause of the slow relaxation. Here we suggest that phase space is uniform at the critical energy.

For the purpose, we calculate Lyapunov exponent for ten initial conditions when $E = E_c$ and $N = 80$. Results are as follows;

$$\begin{aligned}\lambda_1 = & 0.229331, 0.229133, 0.231313, 0.229381, 0.229121, \\ & 0.231387, 0.230692, 0.231396, 0.228699, 0.230357.\end{aligned}\quad (22)$$

That is,

$$\lambda_1 = 0.2300 \pm 0.0014 \quad (E/N = 0.75, N = 80). \quad (23)$$

On the other hand, Table 1 shows standard deviations of local Lyapunov exponents for the critical energy $E_c/N = 0.75$ described in Fig.8. Then we find that the distribution of Lyapunov exponents is narrow enough comparing with the standard deviations of local Lyapunov exponents. Hence initial conditions have no influences on the values of λ_1 , in other words, phase space is uniform for the intensity of instability at the critical energy.

The values of λ_1 are not different between relaxations of power type and exponential type because the first and the second value of Eq.(22) correspond to the initial conditions of Figs.2(b) and 4(a), respectively. Hence we can not distinguish between the two types only from the values of λ_1 , which are time average of instability.

N	standard deviation
40	0.065916
80	0.056067
200	0.055623

Table 1: Standard deviations of local Lyapunov exponents when $E/N = E_c/N = 0.75$. Those probability distributions are described in Fig.8.

3 Summary and discussions

Relaxation processes are numerically investigated through temporal evolutions of local order parameter in a Hamiltonian system which has second order phase transition. Slow relaxations of power type of order parameter are observed at the critical point for some initial conditions. It is the first time that power type decay is observed for order parameter, which is an important quantity to observe the system, from temporal evolutions of equations of motion derived from a Hamiltonian.

To understand the cause of the slow relaxation dynamically, we investigated dynamical properties of the system. Then we found that the slow relaxations occur in highly chaotic systems rather than weakly chaotic systems. That is, mechanism of the slow relaxations can not be explained by theories for perturbed systems, hence we must consider new theories for highly chaotic systems.

The slow relaxations of power type are observed when $N = 40$ and 80 . However we have not observed the slow relaxation when $N > 80$, and some initial conditions yield exponential relaxations even if $E = E_c$.

Why the slow relaxation has not been observed when N is large? Large number of N makes a situation such that to detect slow relaxation is difficult because the larger N is, the clearer the critical point may be, and energy has small errors in calculations although we set $E = E_c$ initially. Moreover we do not know the exact value of the critical energy in the meaning of dynamics. In other words, we do not know whether dynamics also has the same critical energy E_c which is obtained from statistical mechanics. However to obtain exact value of critical energy is difficult since appearance of the slow relaxation depends on initial conditions. We can not say whether the slow relaxation appears in the thermodynamic limit. The author expects the appearance because we have supporting evidences for appearance but do not have for absence.

Appearance of the slow relaxation agrees with a result of phenomenological theory of statistical mechanics; van Hove theory. However, contrary to the phenomenological theory, relaxation processes do not start at once even the system are in non-equilibrium states. That is, we observed an agreement and a disagreement between our results from numerical experiments and results from a phenomenological theory. To research the origin of the agreement is interesting to understand contracted dynamics which yields motion of order parameter.

From temporal evolutions and probability distributions of local Lyapunov exponents, we understood the system is highly chaotic rather than weakly chaotic at the critical energy. Hence we may not apply theories for perturbed systems to understand the slow relaxation of power type at the critical energy. We must make new theories for highly chaotic systems. For the purpose we must understand structure of phase space when the system is highly chaotic and has many degrees of freedom. To understand the structure of phase space, a result is obtained such that the structure of phase space may be uniform for the intensity of instability at the critical energy.

Acknowledgement

I express my thanks to Tetsuro Konishi for useful discussions and a careful reading of the manuscript. I acknowledge helpful discussions with Hiroyasu Yamada and Akira Yoshimori. I wish to thank Kazuhiro Nozaki for special encouragements. I thank referees for fruitful comments.

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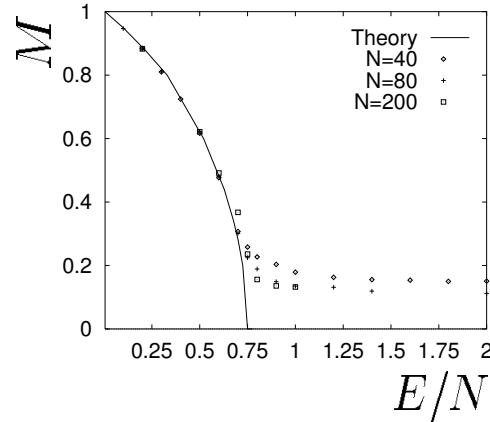


Figure 1: Order parameter M vs. energy per unit particle E/N . $L = 2^{23} \times 0.01$ ($\sim 10^5$, namely 10^7 steps) (cf. Eq.(5)). (\diamond): $N = 40$. (+): $N = 80$. (\square): $N = 200$. Solid line represents a result of theory of statistical mechanics. Results are good agreement with the theory.

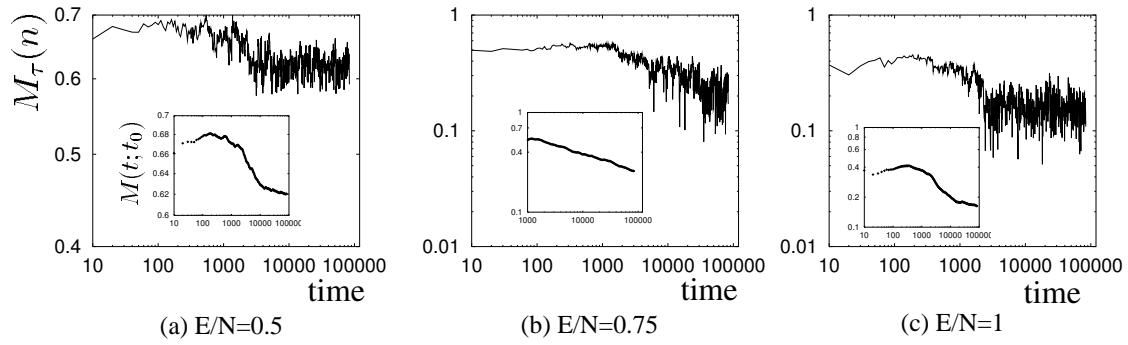


Figure 2: Log-log plotted temporal evolutions of $M_\tau(n)$ (cf. Eq.(7)). $N = 80, \tau = 10$. (a) $E/N = 0.5$. (b) $E/N = E_c/N = 0.75$. (c) $E/N = 1$. Insets: The temporal evolution of $M(t; t_0)$ which is yielded by the time series of $M_\tau(n)$ (cf. Eq.(16)), where $t_0 = 1000$ for (b) and $t_0 = 0$ for (a) and (c). When $E = E_c$ slow relaxation of power type appears from t_0 .

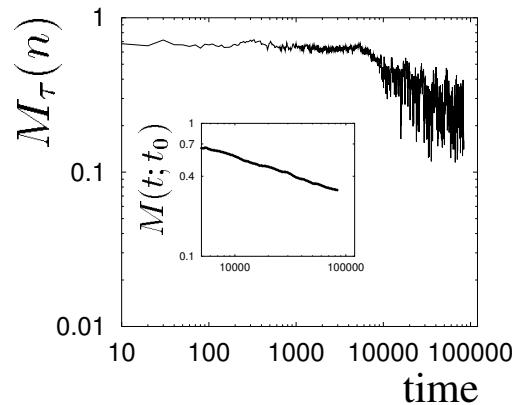


Figure 3: The same with 2(b), but $N = 40, t_0 = 5000$.

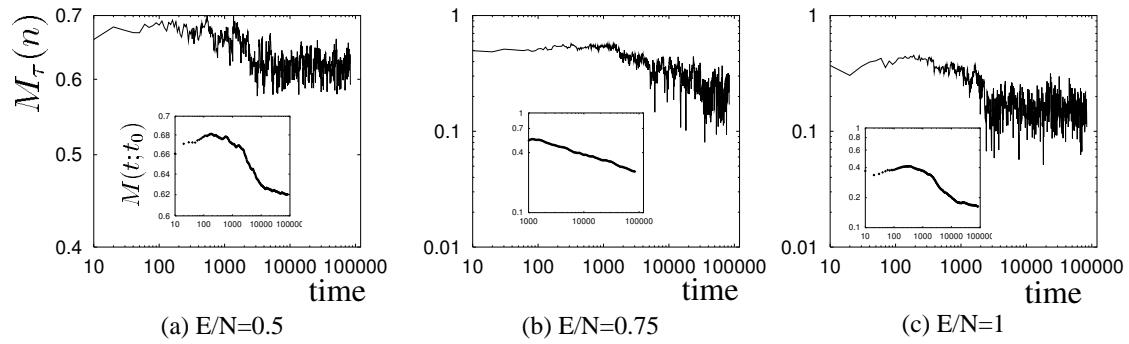


Figure 4: Log-log plotted temporal evolutions of $M_\tau(n)$ (cf. Eq.(7)) at E_c . $\tau = 10$. Insets: Semi-log plotted graphs. (a) $N = 80$. The initial condition is different from one of Fig.2(b). (b) $N = 200$. (c) $N = 1000$. We find relaxations are exponential type rather than power type.

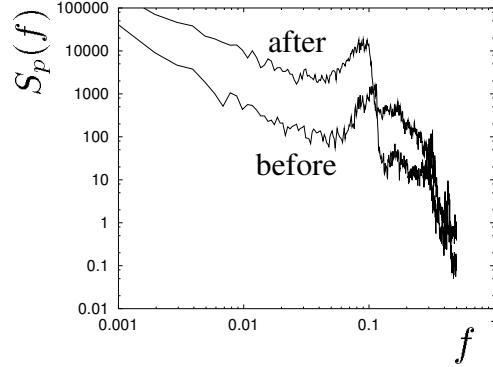


Figure 5: Average of power spectra of momenta p_j 's ($j = 1, 2, \dots, 40$) when $N = 80$ and the initial condition yields slow relaxation of power type. Lower and upper graphs are calculated from time series before ($t \in [0, 1024]$) and after ($t \in [2048, 3072]$) the onset of slow relaxation, respectively. Distinguishing the two graphs, values of the vertical axis are multiplied 10 times for the upper graph. The motion in phase space is not nearly periodic since the graphs do not consist of solitary peaks.

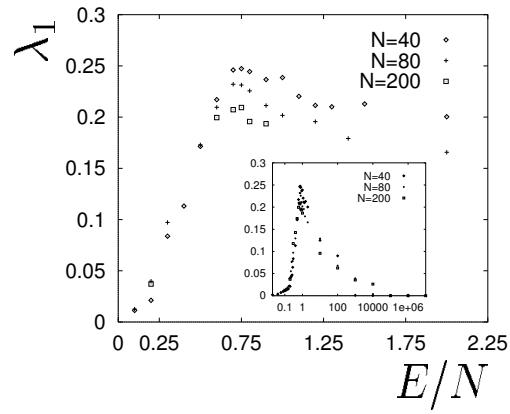


Figure 6: The maximum Lyapunov exponent λ_1 vs. energy per unit particle E/N . $L = 2^{23} \times 0.01$ ($\sim 10^5$, namely 10^7 steps) (cf. Eq.(5)). Inset: The horizontal axis is logarithm scale. $\tau = 0.001$, $L = 2^{20} \times 0.001$ when $E/N \geq 10^3$. (\diamond): $N = 40$. (+): $N = 80$. (\square): $N = 200$. The quantity λ_1 takes the largest values at critical energy, namely $E_c/N = 0.75$, and it goes to zeros in the limit of $E/N \rightarrow 0$ and $E/N \rightarrow \infty$.

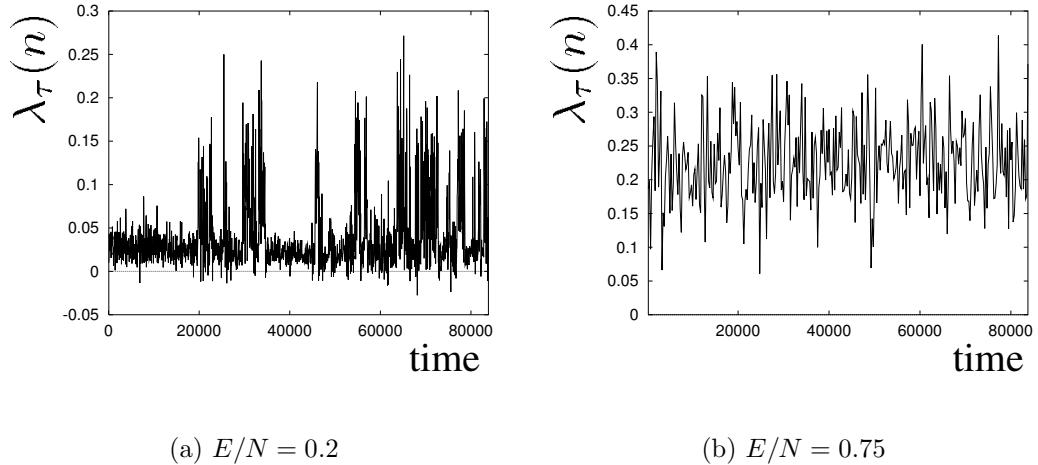


Figure 7: Temporal evolutions of local Lyapunov exponents. $N = 80$. (a): $E/N = 0.2, \tau = 50$. (b): $E/N = 0.75 = E_c/N, \tau = 10$. The initial condition of (b) yields relaxation of power type. Typical intermittency is found in (a), and (b) is not intermittency.

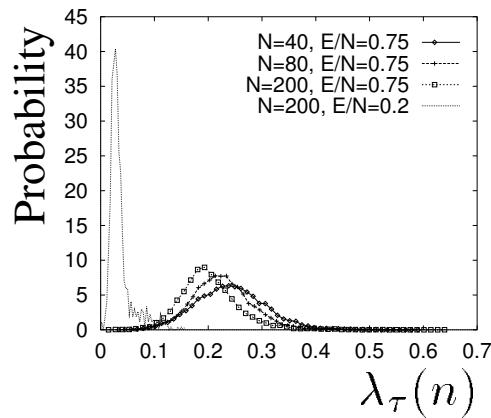


Figure 8: Probability distributions of local Lyapunov exponents. (\diamond): $N = 40$. (+): $N = 80$. (\square): $N = 200$. The value of energy is $E/N = 0.75 = E_c/N$. Dashed line is the case $N = 200, E/N = 0.2$, and intermittency occurs. We set $\tau = 100$ when $E/N = 0.2$. At the critical energy the system is different from intermittency.